Signal Processing on Simplicial Complexes

Feng Ji, Giacomo Kahn and Wee Peng Tay, *Senior Member, IEEE*

Abstract

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I. Introduction

TODO

II. Simplicial complexes

A. Simplicial Complexes...

Simplices are generalisations of triangles to arbitrary dimensions. They are widely studied in algebraic topology since the end of the nintenth century [1] and are now being studied in the light of data analysis. Simplices are points (dimension zero), segments (dimension one), triangles (dimensions two), tetrahedrons (dimension three), etc. Each k-dimensional simplex consists of $k+1$ vertices linked by simplices of lower dimensions. For example, in a tetrahedron, each pair of vertices forms an edge, and each triplet forms a triangle. Any non-empty subset of vertices included in a simplex is said to form a *face* of this simplex, and a k-face whenever its dimension is k. A simplex with $k+1$ vertices is said to be of *dimension* k. In a k-simplex, $(k-1)$-faces are called *facets* of the simplex.

Simplicial complexes are obtained by glueing together simplices with the basic rule that any two simplices of the complex only intersect on simplices of lower dimensions. Additionally, every face of a simplex contained in the complex is also in the complex (the simplicial complex

The authors are with the School of Electrical and Electronic Engineering, Nanyang Technological University, 639798, Singapore (e-mail: jifeng@ntu.edu.sg, giacomo.kahn@ntu.edu.sg, wptay@ntu.edu.sg).

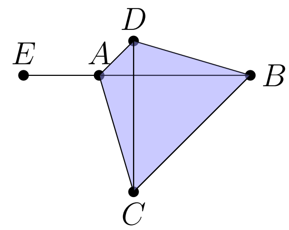

Fig. 1. A simplicial complex. It has two maximal simplices (facets): a 1-dimension edge $\{A, E\}$ and a 4-dimensions tetrahedron $\{A, B, C, D\}$.

is closed by inclusion). A simplicial complex is said to have *dimension* k (and is then denoted simplicial k-complex) if the dimension of its highest dimensional simplex is k. Then, the *facets* of the simplicial complex are simply its maximal simplices.

Figure 1 shows an example of a simplicial complex obtained by glueing an edge and a tetrahedron. The facets of this simplicial complex are $\{A, E\}$ and $\{A, B, C, D\}$.

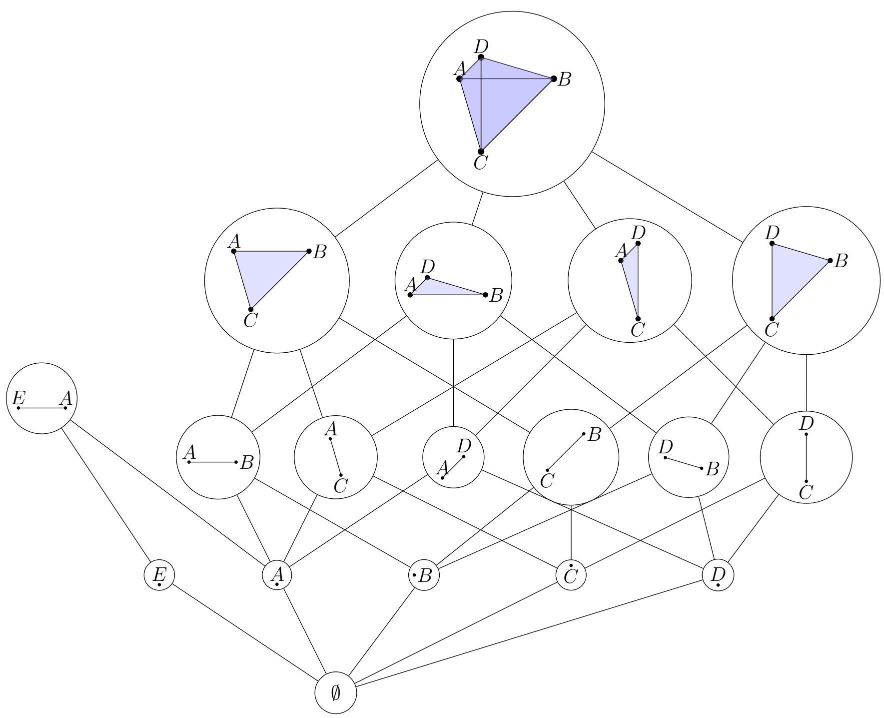

Fig. 2. Simplicial complexes are families of sets closed by inclusion. As so, they can be ordered as meet-semilattices, where the maximal elements are the maximal simplices of the complexe. This shows the meet-semilattice corresponding to our example in Fig. 1. Here, the two maximal elements are the two maximal simplices of Fig. 1. All of the elements of this order, except for the empty set, are themselves simplices.

The ℓ-skeleton of a simplicial k-complex is a representation of this complex with no simplex of dimension higher than ℓ. In our example (Fig. 1), the 2-skeleton of the simplicial complex can be inferred by deleting the tetrahedron $\{A, B, C, D\}$. The maximal simplices are then of dimension two. The 1-skeleton of the same simplicial complex is shown in Fig. 3. In a 1-skeleton, only vertices and edges are shown. The 1-skeleton can be thought of as the underlying graph of the simplicial complex.

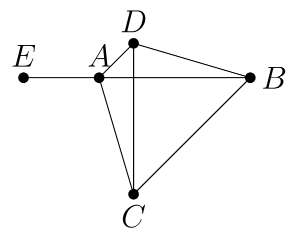

Fig. 3. Faces of dimension higher than 1 are forgotten when dealing with the 1-skeleton.

B. ... And Where to Find Them

1) In theory: Independence systems and more generally accessible systems, which are classes of set systems that contain simplicial complexes, are well known to be useful in the design of efficient enumeration algorithms in data mining applications [2]–[4]. Matroid are simplicial complexes that satisfy an additional condition (the augmentation property). They are widely used as basis for greedy algorithms [5], can be inferred from data [6] and are transversal to lots of fields of computer science and mathematics [7], [8].

2) In practice:

a) Topological Data Analysis: Underlying both traditional Graphs Signal Processing and the use of simplicial complexes in data analysis lurks the very same idea. The idea that the topology (of a graph for GSP, of the local constraints between simplices for simplicial complexes) of the data has meaning, and not only the coordinates of its data points.

Topological data analysis deals with the shape of data by transforming point clouds into simplicial complexes. An overview of TDA can be found in these papers [9], [10] and the TDA community gives numerous ways of creating simplicial complexes from data points, *inter alia* though the use of *lenses* and the MAPPER algorithm. In those approaches, the vertices of the simplicial complexes are usually clusters of data points rather than individual data points.

Figure 4 shows an example of a simplicial complex obtained using MAPPER. A lense in MAPPER is a clustering function that projects data points onto a k-dimensional spaces. Each cluster yield a point on this space. Intersecting clusters form simplices, so that the whole system forms a simplicial complex. By varying the lenses and its parameters, topological data analysts are able to infer new knowledge from the shape of the data. Then, the signal can be drawn from the original dataset, as a mean or any combination of the clustered data points.

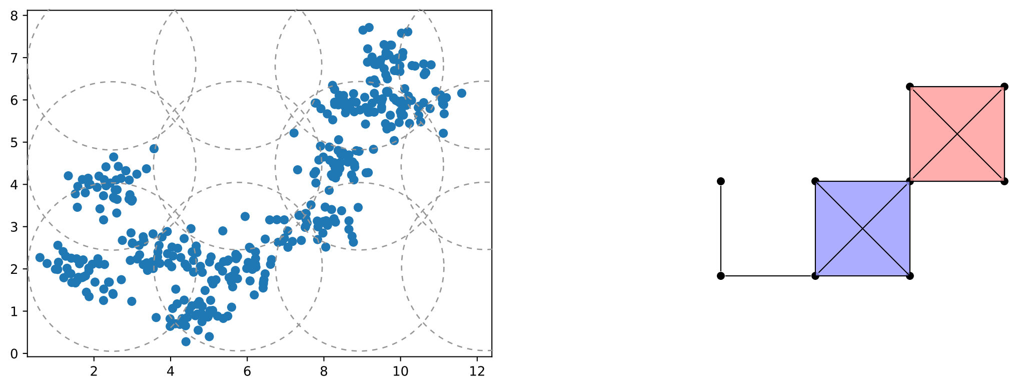

Fig. 4. Left: A data points cloud and a covering with cirlces. Right: Simplicial complex build from the points on the left. Two vertices are linked if the corresponding clusters share points.

b) Smart city and sensor classes: In [11], the authors propose a framework for a smart city based on the Internet of Things (IoT). This framework uses both fixed and mobiles sensors, so the coordinate of the sensors may vary. In such cases, sensors might be linked if they are able to communicate with each other, and higher arity simplices can represent either similar sensors (temperature, sound, pollution), sensors on similar devices (mobile phones, vehicles) or in the same building. This kind of high dimensional modeling allows not only to perform different data analysis but also to consider reprogramming all sensors of a kind at the same time, changing communication protocols for a class of sensors or similar tasks.

c) More generally: hypergraphs: As we will discuss in the next section, our approach can be used directly on hypergraphs. The applications of hypergraphs in data analysis are numerous. Some can be found in this 1995's survey [12] focusing on the problems related to the computation of hypergraph transversals. In biology [13] and bio-chemistry [14] and chemistry [15], hypergraphs are used extensively to model complex interactions between proteins, chemicals or species. An example of high arity relation between proteins is shown in Figure 5.

Other applications can be found surrounding geometric objects [16], image approximation [17], extraction of a skeleton from a data cloud [18] or itemset mining [19] in high dimensional datasets.

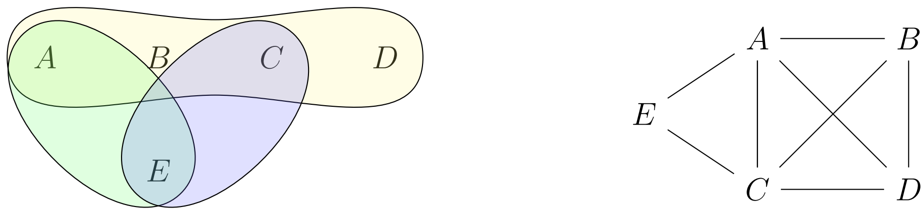

Fig. 5. This hypergraphs models interactions between five proteins. Signals associated with the vertices can be concentration, number, temperature, etc. This figure is inspired by the example in [13]. It shows that a n-ary relation cannot be represented by binary relations without a loss of informations.

III. Generalized Laplacian

Recall that graph signal processing relies heavily on the notion of “shift operator". A popular choice is the graph Laplacian. We start by generalizing this notion.

Definition 1. *Let X be a finite metric space of size $|X| = n$. A generalized Laplacian consists of the following data:*

(A) a weighted, undirected graph $G = (V', E')$,

(B) a set function $f : X \to V'$, and

(C) A linear transformation $T : \mathbb{R}^{|X|} \to \mathbb{R}^{|V'|}$,

such that the following holds:

(a) f is one-one,

(b) the $f(v)$ component $T(x)$ is the same as the v component of v for each $v \in X$ and $x \in \mathbb{R}^{|X|}$, and

(c) the sum of each row of T is a constant.

Let L_G be the Laplacian of the weighted graph G. The generalized Laplacian associated with the data (G, f, T) *is defined as*

$$L_{(G,f,T)} = T' \circ L_G \circ T : \mathbb{R}^{|V|} \to \mathbb{R}^{|V|}.$$

We may abbreviated $L_{(G,f,T)}$ by L_X if no confusion arises from the context.

We give some intuitions on Definition 1. We require that f is one-one to ensure that f “embeds" X in G such that we may perform the shift operation on G. Condition (b) and (c) on T is saying roughly that the signal on $v \in X$ is preserved at its image of f in G, otherwise an averaging process takes place.

Lemma 1. *(a) L_X is symmetric.*

(b) L_X is positive semi-definite.

(c) Constant signals are in the 0 eigenspace of L_X. The 0-eigenspace E_0 of L_X is 1-dimensional if and only if G is connected.

Proof: (a) L_X is symmetric because G is assumed to be undirected and hence L_G is symmetric.

(b) Similar to (a), L_X is positive semi-definite because L_G is positive semi-definite.

(c) As we assume that the sum of each row of T is a constant, therefore if x is a constant vector, then so is $T(x)$. Since constant vectors are in the 0-eigenspace of L_G, we have $L_G \circ T(x) = 0$ and so is $L_X(x)$. Therefore, the dimension of E_0 is at least 1.

Now assume that x is in E_0. Therefore,

$$0 = \langle x, L_X(x) \rangle = \langle T(x), L_G \circ T(x) \rangle.$$

Consequently, $T(x)$ belongs to the 0-eigenspace of L_G, which is 1-dimensional if and only if L_G is connected.

By Condition (b), the operator T is injective. Therefore, E_0 is 1-dimensional if and only if $T(E_0)$ is 1-dimensional, which is in turn equivalent to G being connected as we just observe. ■

By Lemma 1, the generalized Laplacian L_X enjoys a few desired properties. In particular, being symmetric permits an orthonormal basis consisting of eigenvectors of L_X. Therefore, one can devise a Fourier theory analogous to traditional GSP. Moreover, as L_X is positive semi-definite, we may perform smoothness based learning. The constant vectors belong to the 0-eigenspace is also desirable as it agrees with the intuition that “constant signals are smoothest".

On the other hand, Lemma 1 asserts that L_X is indeed very similar to the Laplacian of a graph. The theory will be less useful if we are only able to produce weighted graph Laplacian, which we shall prove to be untrue. To this point, we introduce the following notion.

Definition 2. *We call L_X is of* graph-type *if all the diagonal entries of L_{A_X} are non-negative and all the off-diagonal entries are non-positive.*

Now we are going to give an explicit construction of L_X together with the choice of G, f and T. As an implicit requirement, we would like the construction to recover the usual Laplacian if X is a graph.

For the simplest case, assume $X \cong \Delta_n$ is a weighted n-simplex, i.e., X is homeomorphic to the standard n-simplex and its 1-skeleton X^1 is a weighted graph. We label the vertices of X by $v_1, \dots, v_{n+1}$. The graph $G_X = (V, E)$ is constructed as follows: $V = \{v_1, \dots, v_{n+1}, u\}$ with a single additional vertex u, which is understood as the barycenter of X. There is no edge between v_i and v_j for any pair $1 \leq i \neq j \leq n+1$. On the other hand, $(v_i, u) \in E$ for each $1 \leq i \leq n+1$.

The edge weight $w(v_i, u)$ of (v_i, u) is computed as follows:

$$w(v_i, u) = \frac{1}{\binom{n}{2}} \Big(\sum_{v_i \neq v_j \neq v_k \neq v_i} (v_j, v_k)_{v_i} \Big),$$

where $(v_j, v_k)_{v_i}$ is the Gromov product defined in ??.

Illustrations for $X \cong \Delta_2$ and $X \cong \Delta_3$ are shown in Figure 6.

We have a canonical choice for T: $T(v_i) = v_i$. For f, it is identity on the v_i components, while the average is assigned to the u component. In the matrix form,

$$f = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 1/(n+1) & 1/(n+1) & \dots & 1/(n+1) \end{bmatrix}.$$

It is straightforward to verify (G_X, T, f) verifies the condition of Definition 1. Thus, we have an associated generalized Laplacian L_X.

For a general finite X, we have a decomposition X as the union of maximal simplexes and the generalized Laplacian

$$L_X = \sum_{\sigma} L_{\sigma},$$

where the summation is taken over all maximal simplexes of X.

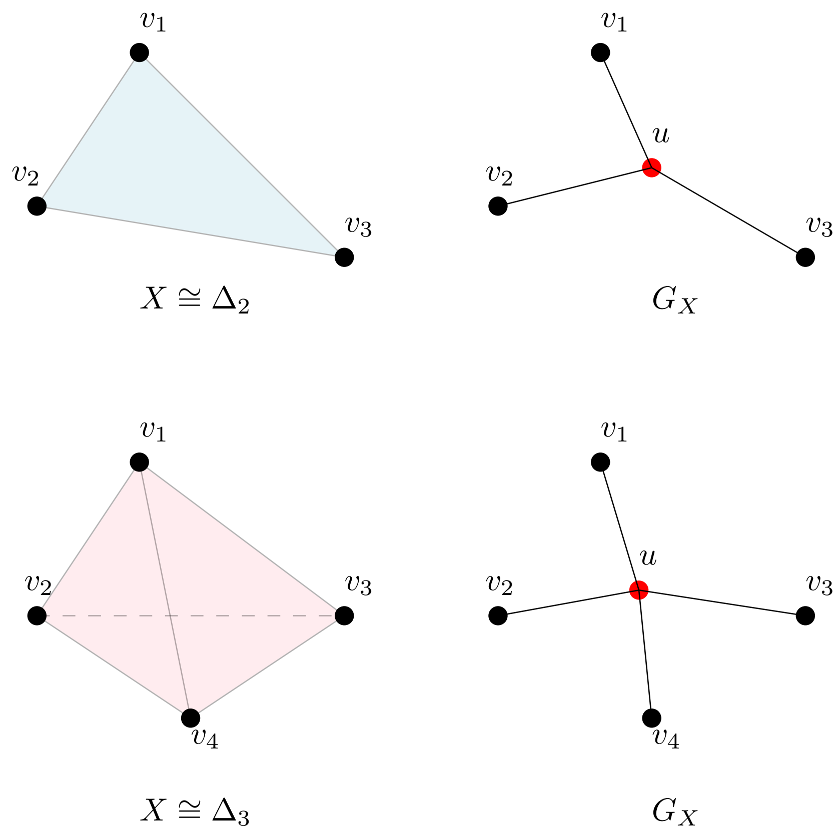

Fig. 6. Graphical illustration of the shape of G_X for $X \cong \Delta_2$ and $X \cong \Delta_3$.

To give some insights of the construction, we notice that for $X \cong \Delta_n$, it is topologically a point (more precisely, homotopy equivalent to a point). Therefore, if we want to approximate X by a graph G_X that preserves this topological property, G_X must be a tree. In addition, if we do not want to break the symmetry of the vertices, the most natural step to do so is to add one additional node (the barycenter) connected to every vertex in the original graph. The edge weights of G_X are chosen to approximate the metric of X as close as possible.

Other evidence for the construction shall be discussed in subsequent sections. We end this section with the example when X itself is a graph.

Example 1. *As a warm-up, we consider the case that X is itself a graph. The maximal simplexes are just edges. For an edge $e = (v_1, v_2)$ with weight w. The associated graph G_e contains 3 nodes, v_1, v_2 and an additional node u. As there are only two indices, the edge weight is computed slightly differently as $w(v_i, u) = (v_{3-i}, v_{3-i})_{v_i} = w$. Hence the generalized Laplacian is computed* *as*

$$L_e = \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{bmatrix} \begin{bmatrix} w & 0 & -w \\ 0 & w & -w \\ -w & -w & 2w \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} w & -w \\ -w & w \end{bmatrix}.$$

This recovers the usual Laplacian for the edge e*. Summing over all the edges, we see that the generalized Laplacian is the same as the usual Laplacian.*

IV. 2-COMPLEXES

In this section, we focus on 2-complexes, on which the main applications base on. For a weighted to simplex $X \cong \Delta_2$, assume that the edge weights are $w(v_1, v_2), w(v_1, v_3)$ and $w(v_2, v_3)$. The edge weights of G_X are

$$a = (v_2, v_3)_{v_1} = (w(v_1, v_3) + w(v_1, v_2) - w(v_2, v_3))/2,$$

$$b = (v_1, v_3)_{v_2} = (w(v_2, v_3) + w(v_1, v_2) - w(v_1, v_3))/2,$$

$$c = (v_1, v_2)_{v_3} = (w(v_1, v_3) + w(v_2, v_3) - w(v_1, v_2))/2.$$

If the edge weights satisfy the triangle inequality, then $a \geq 0, b \geq 0, c \geq 0$. Conversely, given $a \geq 0, b \geq 0, c \geq 0$, we are able to recover the edge weights by taking pairwise sums.

The generalized Laplacian L_X is thus given by:

$$L_X = \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \end{bmatrix} \begin{bmatrix} a & 0 & 0 & -a \\ 0 & b & 0 & -b \\ 0 & 0 & c & -c \\ -a & -b & -c & a+b+c \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$
$$= \frac{1}{9} \begin{bmatrix} b+c+4a & c-2a-2b & b-2a-2c \\ c-2a-2b & a+c+4b & a-2b-2c \\ b-2a-2c & a-2b-2c & a+b+4c \end{bmatrix}.$$

Definition 3. *Define the* shape constant γ_X *of* X *as*

$$\gamma_X = \min\{\frac{5w(v_i, v_j) - w(v_i, v_k) - w(v_j, v_k)}{2}, \{i, j, k\} = \{1, 2, 3\}\}.$$

In general, γ_X can be negative. This happens when there is at least one very short edge. We use it to address an issue left over from the previous section.

Lemma 2. *Suppose $X \cong \Delta_2$ is a 2-simplex. Then L_X is of graph type if and only if the $\gamma_X \geq 0$.*

Proof: A direct computation shows that

$$-\gamma_X = \max\{c - 2a - 2b, b - 2a - 2c, a - 2b - 2c\}.$$

As the diagonal entries of L_X are all positive, it is of graph type if and only if $-\gamma_X \leq 0$, i.e., $\gamma_X \geq 0$. ■

In the case of 2-simplex, we may also give the following interpretation of L_X with the graph Laplacian L_{X^1} of the 1-skeleton X^1.

Consider a graph signal $x = (x_1, x_2, x_3)'$ on the vertices $\{v_1, v_2, v_3\}$. Let y be the first order difference $(x_3-x_2, x_1-x_3, x_2-x_1)'$. By a direct computation, one observes that L_X is determined by

$$9\langle x, L_X(x)\rangle = \langle y, L_{X^1}(y)\rangle.$$

It says L_X is a higher order difference, though the point-of-view cannot be generalized beyond dimension 2.

If X is a general 2-dimensional simplicial complex, the Laplacian L_X takes contribution form Laplacian of 2-simplexes computed as above and usual edge Laplacians. We next study spectral properties of L_X. In particular, we want to compare L_X and L_{X^1} as the latter is well-studied. Recall that $A \preceq B$ if $B - A$ is positive semi-definite.

Lemma 3. *Suppose X is a finite 2-dimensional simplicial complex with each edge of length 1. Let $k_{\max}$ and $k_{\min}$ be the largest and smallest numbers of 2-simplexes can share a single edge. Then*

$$\max\{\frac{k_{\min}}{3}, \frac{1}{3}\} \cdot L_{X^1} \preceq L_X \preceq \frac{k_{\max}}{3} L_{X^1}.$$

Proof: We sketch the main idea of the proof. It suffices to show $L = L_X - \max\{\frac{k_{\min}}{3}, \frac{1}{3}\} \cdot L_{X^1}$ or $L = \frac{k_{\max}}{3} L_{X^1} - L_X$ is the Laplacian of a (possibly disconnected) graph. For this, one only need to compute the off-diagonal entries of L and show they are non-positive, which follows from easy computation. ■

If X is a 2D-mesh (triangulation) of a compact 2-manifold, then $k_{\min} = 1$ and $k_{\max} = 2$. This is because at most two 2-simplexes can share a common edge and along the boundary each edge is contained in a single 2-simplex.

Recall that a filter F is *shift invariant* w.r.t. L_{X^1} if $F \circ L_{X^1} = L_{X^1} \circ F$. If the graph Laplacian L_{X^1} does not have repeated eigenvalues, then F is shift invariant if and only if $F = P(L_{X^1})$ for some polynomial P of degree at most $n-1$. The shift invariant family is of particular interest and they are readily estimated as one only has to learn the polynomial coefficients. Due to this fact, L_X will be less interesting if it is shift invariant w.r.t. L_{X^1}, e.g., when X is a single 2-simplex with equal edge weights (more examples are shown in Figure 7). However, this does not happen in general.

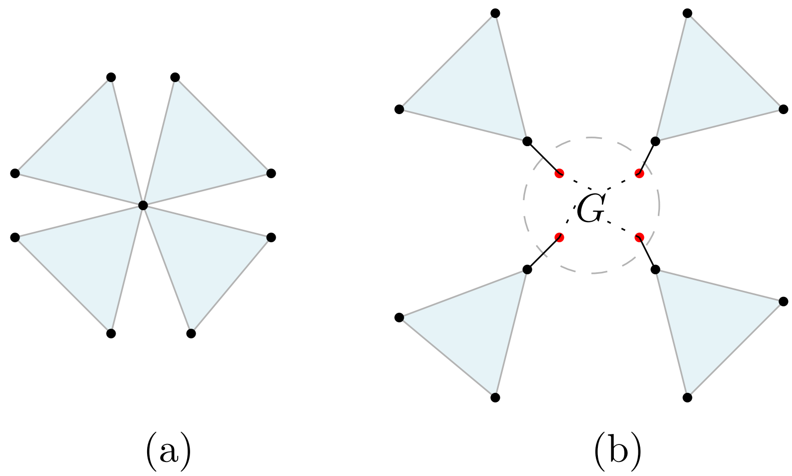

Fig. 7. In (a), if all the edge weights are the same then $L_X = 1/3 L_{X^1}$ is shift invariant w.r.t. L_{X^1}. However, in (b), as long as the 4 red nodes are contained in a graph G (at the center), then L_X is not shift invariant w.r.t. L_{X^1} by Proposition 1, even if we allow arbitrary positive edge weights.

Proposition 1. *Suppose X is a 2-complex such that the following holds (illustrated in Figure 8):*

(a) In X, any two 2-simplexes are not connected by an direct an edge.

(b) In X, if a vertex v is not contained in any 2-simplex, then it is connected to at most one 2-simplex. There is at least one such vertex.

(c) Each edge is contained in at most one 2-simplex.

Then L_X is not shift invariant w.r.t. L_{X^1}.

Proof: The proof is given in Appendix A. ∎

V. Learning simplicial complex from signals on a point cloud

In this section, we discuss the approach to learn a 2-complex structure X given the 1-skeleton X^1, or just the set of vertices X^0, by probably using signals on X^0.

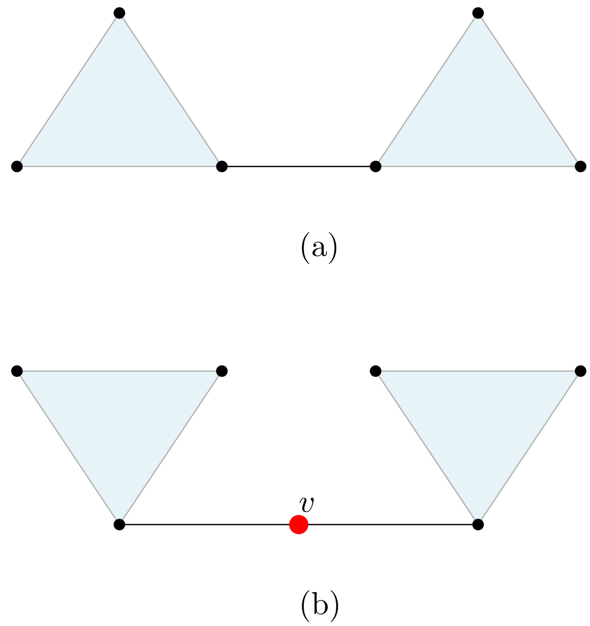

Fig. 8. Illustration of the two situation disallowed by the first two conditions of Proposition 1.

If X^1 is an unweighted graph, we assign weight 1 to each edge. Otherwise, If pairwise similarities of X^0 are given, then we define the weights between (v_1, v_2) to be the inverse of the similarity (i.e., we want two nodes to be closer if they are more similar).

The general idea goes as follows. We first identify all the set Δ_{X^0} of all possible 2-simplexes. Depending on the problem, there are two main cases:

(a) If X^1 is given, then triple nodes (v_1, v_2, v_3) belongs to Δ_{X^0} if and only if $(v_1, v_2), (v_1, v_3)$ and (v_2, v_3) are all edges of X^1.

(b) If only X^0 is given, then we assums Δ_{X^0} contains any triple (v_1, v_2, v_3) of distinct nodes in X^0.

Given two non-negative numbers $r_1 \leq r_2$, we define $\Delta_{X^0}(r_1, r_2)$ to be the subset of Δ_{X^0} consisting of triples (v_1, v_2, v_3) whose pairwise edge weights are within the closed interval $[r_1, r_2]$. Apparently, if $r_1' \leq r_1 \leq r_2 \leq r_2'$, then $\Delta_{X^0}(r_1, r_2) \subset \Delta_{X^0}(r_1', r_2')$. Hence, we have the fundamental filtration $\emptyset = \Delta_{X^0}(0,0) \subset \Delta_{X^0}(0,r) \subset \Delta_{X^0}(0,r') \subset \Delta_{X^0}(0,\infty) = \Delta_{X^0}$ for $r \leq r'$.

For the general framework, we would like perform the following steps:

(a) Order all the 2-simplexes of Δ_{X^0} in a queue Q:

(i) Choose $r_0 = 0 \leq r_1 \leq \ldots \leq r_n$ such that $\Delta_{X^0} = \Delta_{X^0}(0, r_n)$. A 2-simplex in

$\Delta_{X^0}(r_i, r_{i+1})$ is always ordered before $\Delta_{X^0}(r_{i+1}, r_{i+2})$.

(ii) We order the 2-simplexes of $\Delta_{X^0}(r_i, r_{i+1})$ in such a way that 2-simplexes sharing many common edges are ordered later in the queue, and we shall be more explicit later on.

(b) Partition Q as a disjoint union $Q = \cup_{1 \leq i \leq p} Q_i$ such that there size are approximately uniform and Q_i takes the element of Q ranging from $1 + \sum_{1 \leq j < i} ||Q_j|$ to $\sum_{1 \leq j \leq i} ||Q_j|$.

(c) Let X_0 be $X^0 \cup X^1$. For each $1 \leq i \leq p$, we construct a 2-complex X_i be adding the 2-simplexes of Q_i (and the associated edges) to X_{i-1}. Form the associated generalized Laplacians $L_i = L_{X_i}$.

(d) Form convex combinations, $L_{i,t} = tL_i + (1-t)L_{i+1}$. Learn the best i and t combination from signals such as learning the best polynomial filter to fit data. The parameter i is discrete, which can be only be learnt from exhausted search, while for each fixed i, t can be learnt from continuous optimization.

For completeness, we describe the algorithm for Step (ii) (illustrated in Figure 9):

(a) For each i, randomly order the 2-simplexes of Q_i to form Q.

(b) For j ranges from the 2-simplexes of Q (following the ordering), assume we have ordered $x_1, \ldots, x_j$. Search for the rest of the 2-simplexes of Q. If x is sharing a common edge with x_j, re-order Q by placing x at the end of Q. Once all $x \in Q \backslash \{x_1, \ldots, x_j\}$ is gone through once, repeat the procedure for x_{j+1}.

Now, we describe in more details on how to use the approach to learn 2-complex structure of X^0 with signals on X^0. In addition, we also want to discuss filter learn. The basic form of the problem is specified as follows: there are two sets of signals f_1, f_2 on X^0. Learn the structure of X and an appropriate filter F that such that $f_2 = F(f_1) + g$, where g is the white noise. We propose to solve the following optimization problem:

$$\min_{1 \leq i \leq p} \min_{\substack{t \in [0,1] \\ (a_0, \ldots, a_b) \in \mathbb{R}^{b+1}}} \left\| \left(\sum_{1 \leq j \leq b} a_j L_{i,t}^j \right)(f_1) - f_2 \right\|^2, \tag{1}$$

where b is a pre-determined bound on the degree of the polynomial.

VI. Simulation results

Experiments:

2-simplex reconstruction

Outlier detection (malfunction)

Signal compression

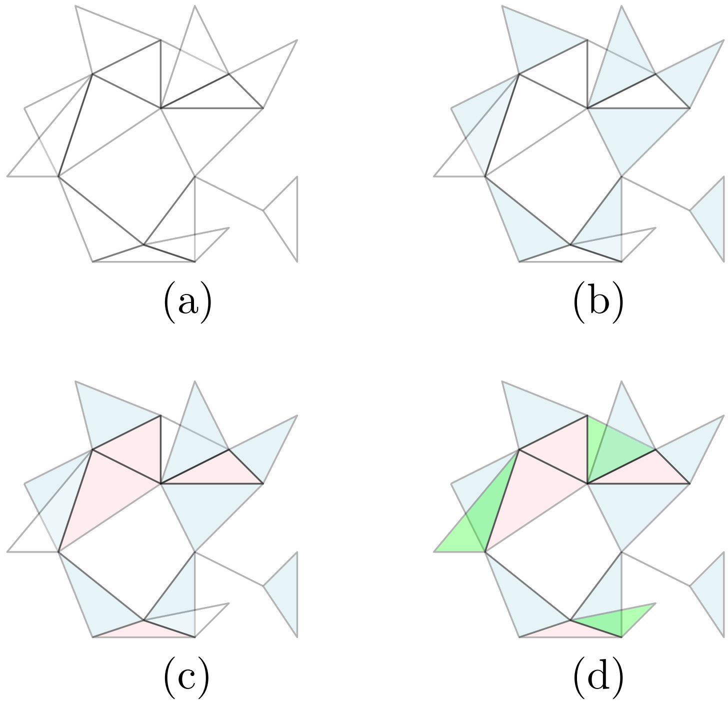

Fig. 9. In this example, the blue 2-simplexes in (b) are (randomly) ordered first in Q. After which, we have the pink 2-simplexes in (c). Finally, the green 2-simplexes are ordered last in Q.

Appendix A
Non shift invariance

In this appendix, we assume X is a 2-complex of size n and discuss conditions ensure L_X is not shift invariant w.r.t. L_{X^1}. We are mainly interested in geometric conditions, which can be observed directly from the shape of X. As a corollary, we prove Proposition 1.

For convenience, we introduce the following notion.

Definition 4. *If a matrix M is the Laplacian of a weighted graph G, then we say M is of graph type G. Moreover, we call X has* distinctive 2-simplexes *if (a) either $L_X - L_{X^1}$ or $L_{X^1} - L_X$ is of graph type G; and (b) an edge $e = (v_1, v_2)$ of G has positive edge weights when e belongs to a 2-simplex of X.*

Lemma 4. *X has distinctive 2-simplexes if either of the following holds:*

(a) $k_{\max} \leq 1$, i.e., each edge is contained in at most one 2-simplex.

(b) $k_{\max} \leq 2$ and all the edges have weight 1.

(c) $k_{\min} \geq 4$ and all the edges have weight 1.

Proof: (a) As any constant vector is in the kernel of $L = L_{X^1} - L_X$, the sum of each row is 0. If (i,j) is an edge of X not contained in any 2-simplex, then the (i,j)-th entry of L is 0. It suffices to show that if (v_i, v_j) is any edge contained in a 2-simplex, then the (i,j)-th entry of L is negative. Let $a > 0$ be the weight of (v_i, v_j) and $b > 0, c > 0$ be the weights of the other two edges of the 2-simplex containing (v_i, v_j). A direct calculation shows that the (i,j)-the entry of L is $-(13a + b + c)/18 < 0$.

(b) and (c) can be shown by the same argument by considering $L_{X^1} - L_X$ and $L_X - L_{X^1}$ respectively. ∎

Assume for the rest of this section that X has distinctive 2-simplexes. We want to study common vectors of both L_{X^1} and L_X. To this end, we divide the discussion into two parts: for such an eigenvector, whether the associated eigenvalues are the same or different.

Definition 5. *We call a vertex v is 1-interior if it is not contained in any 2-simplex and 2-interior if each edge containing v belongs to a 2-simplex (see Figure 10 for an example).*

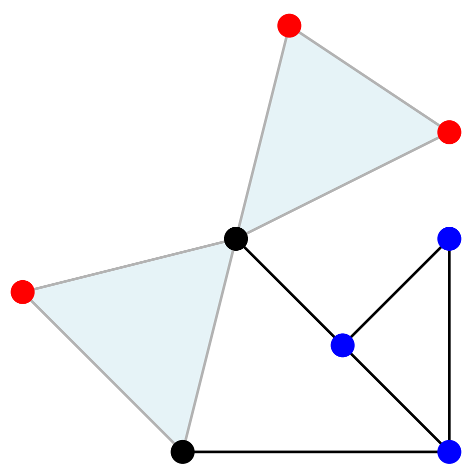

Fig. 10. In this example, all the blue nodes are 1-interior and red nodes are 2-interior. Hence, for the parameters in Lemma 5(b) $m_1 = 3$ and $m_2 = m_3 = 1$. Moreover, in Lemma 6(b), $m_4 = 2$ counts the two black nodes.

Lemma 5. *(a) Let K be the vector space spanned by common eigenvectors with the same eigenvalue of L_X and L_{X^1}. Then K is a space of $\ker(L_X - L_{X^1})$.*

(b) Let m_1 be the number of 1-interior nodes of X, m_2 be the number of connected components of smallest complex containing all the 2-simplexes of X, and m_3 be the number of such components containing some 2-interior nodes. Then $\dim K \le m_1 + m_2 - m_3$.

Proof: (a) As we assume that X has distinctive 2-simplexes, $L_{X^1} - L_X = L_G$ or $-L_G$ for some graph G whose positive edge weights are supported on 2-simplexes of X. Therefore, if w is a common eigenvector with the same eigenvalue, then $L_G(x) = 0$, i.e., $w \in \ker(L_G)$. As $\ker(L_G)$ is a vector space, K as spanned by these v's is also contained in $\ker(L_G)$.

(b) Notice that $\ker(L_G)$ is the same as the number of connected components of G. The set of connected components of G consists of: (1) each 1-interior node of X is an isolated component of G, (2) a union of 2-simplexes that is connected. They are of size m_1 and m_2 respectively. Suppose a component C of the second type contains a 2-interior node v and w is a common eigenvector with the same eigenvalue $\lambda > 0$. Then $L_{X^1}(w)(v) = L_G(w)(v) = 0$. However, $L_{X^1}(w)(v) = \lambda w(v)$, and hence $w(v) = 0$. Hence, w is 0 are all of C as w is constant on C. Hence, the vectors of K vanishes on such a C. Therefore, $\dim K \leq m_1 + m_2 - m_3$. ■

Now we consider common eigenvectors of L_{X^1} and L_X with different eigenvalues.

Lemma 6. *Suppose w is a common eigenvectors of L_{X^1} and L_X with different eigenvalues. Then*

(a) w is 0 at 1-interior nodes of X.

(b) If v belongs to a 2-simplex and any 1-interior neighbor of v is not connected to any other node belonging to a 2-simplex, then w is 0 at v. Denote the number of such vertices by m_4.

Proof: Suppose the eigenvalues of w are $\lambda_1 \neq \lambda_2$.

(a) Let v be a 1-interior node. Then $L_X(w)(v) = L_{X^1}(w)(v)$ as the neighborhood of v in X and X^1 are identical. This implies that $\lambda_1 w(v) = \lambda_2 w(v)$. This is possible only if $w(v) = 0$.

(b) Let v' be a 1-interior neighbor of v. By (a), $w(v') = 0$. As v' is not connected to any other node belonging to a 2-simplex, w is 0 at the neighbors of v' except at v. Hence $0 = L_{X^1}(w)(v') = aw(v)$, where a is the positive edge weight between v and v'. This proves (b). ■

Now, we are ready to state and prove the main result of this section.

Theorem 1. *If $\dim K \leq m_1+m_4 < n$, then there does not exist any orthonormal basis consisting of common eigenvectors of both L_{X^1} and L_X. In particular, this holds if $m_2 \leq m_3 + m_4$.*

Proof: Suppose on the contrary that $W = \{w_1, \ldots, w_n\}$ (column vectors) is an orthonormal basis consisting of common eigenvectors of L_{X^1} and L_X. There are at most $\dim K$ vectors of W each shares the same eigenvalue. Without loss of generality, assume they are $\{w_1, \ldots, w_{\dim K}\}$

and let w_1 be the constant vector $(1/\sqrt{n}, \ldots, 1/\sqrt{n})'$. Moreover, by re-indexing, we further assume that the first $m_1 + m_4$ indices correspond to the set S of 1-interior nodes and nodes satisfy Lemma 6(b).

By abuse of notation, write W for the $n \times n$ matrix whose i-th column being w_i. As the columns of W forms a orthonormal basis, so do the rows of W. On the other hand, by Lemma 6, only the leading $(m_1 + m_4) \times \dim K$ block W_1 of the first $m_1 + m_4$ rows of W can contain non-zero entries. Hence, the rows of W_1 forms an orthonormal system. This shows that $m_1 + m_4 \leq \dim K$.

We claim $m_1 + m_4 \neq \dim K$. For otherwise, W_S is an $\dim K \times \dim K$ matrix with orthonormal rows. Hence, the columns of W_S is also an orthonormal system. However, this is impossible as the norm of the first column of W_S is $\dim K/n < 1$.

Therefore, we have shown that $m_1 + m_4 < \dim K$ with the existence of W. This contradicts the assumption that $\dim K \leq m_1 + m_4$. Furthermore, the condition $m_2 \leq m_3 + m_4$ implies that $\dim K \leq m_1 + m_4$ by Lemma 5(b). ■

As a corollary, we can prove Proposition 1 by counting. First of all, by Condition (c), X has distinctive 2-simplexes. In order to show L_X is not shift invariant w.r.t. L_{X^1}, we want to prove that they cannot have a common orthonormal eigenbasis. By Theorem 1, it suffices to show that $m_2 \leq m_4$ under the assumptions of Theorem 1. Let C be a connected union of 2-simplexes contributing 1 to m_2 in X. In C, there is at least one vertex v_C connected to a 1-interior point for otherwise, we can either add another 2-simplex to enlarge C or X contains no 1-interior point, which contradicts Condition (b). Moreover, v_C cannot be shared by another connected union of 2-simplexes by Condition (a). In conclusion, $C \mapsto v_C$ is an one-one map and hence $m_2 \leq m_4$, and Proposition 1 follows.

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